Resit Exam — Ordinary Differential Equations (WIGDV–07)

Thursday 1 February 2018, 14.00h–17.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (12 points)

Solve the following initial value problem:

$$x^2 y' = (x - y)y, \quad y(1) = 1.$$

Problem 2 (2 + 5 + 6 = 13 points)

Consider the following differential equation:

$$(y^2 + xy + 1)\,dx + (x^2 + xy + 1)\,dy = 0.$$

(a) Show that the equation is *not* exact.

(b) Compute an integrating factor of the form $M(x, y) = \phi(xy)$.

(c) Compute the general solution in implicit form.

Problem 3 (4 + 7 + 9 = 20 points)

Consider the following initial value problem:

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{b}(t), \qquad \mathbf{y}(\tau) = \boldsymbol{\eta}, \tag{$*$}$$

where $A(t)$ is a $n \times n$ matrix.

(a) When do we call $Y(t)$ a fundamental matrix for the homogeneous equation?

(b) Use variation of constants to prove that the solution of $(*)$ is given by

$$\mathbf{y}(t) = Y(t)Y(\tau)^{-1}\boldsymbol{\eta} + Y(t)\int_\tau^t Y(s)^{-1}\mathbf{b}(s)\,ds.$$

(c) Compute a *real* fundamental matrix in the case $A = \begin{bmatrix} -5 & 15 \\ -3 & 7 \end{bmatrix}$.

Problem 4 (10 + 4 + 6 = 20 points)

The space $C([0,1]) = \{y : [0,1] \to \mathbb{R} \,:\, y \text{ is continuous}\}$ provided with the norm

$$\|y\| = \sup_{x\in[0,1]} |y(x)|e^{-\alpha x} \quad \text{where} \quad \alpha > 0,$$

is a Banach space. Consider the following operator:

$$T : C([0,1]) \to C([0,1]), \quad (Ty)(x) = \int_0^x f(y(t))\,dt,$$

where the function $f : \mathbb{R} \to \mathbb{R}$ satisfies:

$$|f(s) - f(t)| \leq \beta|s-t| \quad \text{for all} \quad s,t \in \mathbb{R} \qquad (\beta > 0).$$

(a) Prove that for all $y, z \in C([0,1])$ we have

$$\|Ty - Tz\| \leq \beta L\|y - z\| \quad \text{where} \quad L = \sup_{x\in[0,1]} \int_0^x e^{\alpha(t-x)}dt.$$

(b) Formulate Banach's fixed point theorem.

(c) Prove that there exists a value of $\alpha > 0$ such that Banach's fixed point theorem can be applied.

Problem 5 (10 points)

Compute the general solution of the following 3rd order equation:

$$2u''' + 6u'' - 18u' + 10u = 16 + 26t - 10t^2.$$

Problem 6 (5 + 5 + 5 = 15 points)

Consider the following semi-homogeneous boundary value problem:

$$xu'' + u' + \lambda u = f(x), \quad x \in [1,e], \quad u(1) = 0, \quad u(e) = 0,$$

where $\lambda \in \mathbb{R}$ is a parameter and $f : [1,e] \to \mathbb{R}$ is a continuous function.

(a) Show that $\lambda = 0$ is *not* an eigenvalue of the homogeneous boundary value problem.

(b) Compute for $\lambda = 0$ the Green's function $\Gamma(x,\xi)$.

(c) Solve the boundary value problem for $\lambda = 0$ and $f(x) = 1$.

End of test (90 points)

Solution of problem 1 (12 points)

We can rewrite the equation as

$$y' = \left(1 - \frac{y}{x}\right)\frac{y}{x}.$$

Using the transformation $u = y/x$ gives $y = xu$ so that

$$u + xu' = (1-u)u \quad \Leftrightarrow \quad u' = -\frac{u^2}{x}.$$

(4 points)

The equation for u can be solved by separation of variables:

$$\int -\frac{1}{u^2}\,du = \int \frac{1}{x}\,dx \quad \Rightarrow \quad \frac{1}{u} = \log|x| + C \quad \Rightarrow \quad u = \frac{1}{\log|x| + C}.$$

(4 points)

Hence, the solution of the differential equation for y is

$$y = \frac{x}{\log|x| + C}.$$

(2 points)

The initial condition $y(1) = 1$ gives $C = 1$.
(2 points)

Solution of problem 2 (2 + 5 + 6 = 13 points)

(a) Let $g = y^2 + xy + 1$ and $h = x^2 + xy + 1$, then $g_y = 2y + x$ and $h_x = 2x + y$. Since $g_y \neq h_x$ the equation is not exact.
(2 points)

(b) The function $M(x, y) = \phi(xy)$ is an integrating factor if and only if

$$\begin{aligned} (\phi g)_y = (\phi h)_x \quad &\Leftrightarrow \quad \phi(xy)g_y + x\phi'(xy)g = \phi(xy)h_x + y\phi'(xy)h \\ &\Leftrightarrow \quad \phi'(xy) = \frac{h_x - g_y}{xg - yh}\phi(xy) \\ &\Leftrightarrow \quad \phi'(xy) = \phi(xy). \end{aligned}$$

An obvious solution to this differential equation is $\phi(xy) = e^{xy}$.
(5 points)

(c) Define the function

$$F(x, y) = \int \phi(x, y)g(x, y)\, dx = \int e^{xy}(y^2 + xy + 1)\, dx = (x + y)e^{xy} + C(y).$$

Now we have $F_x = \phi g$ by construction.
(3 points)

Demanding that $F_y = \phi h$ gives

$$e^{xy}(x^2 + xy + 1) + C'(y) = e^{xy}(x^2 + xy + 1),$$

which implies that $C(y)$ has to be a constant function. For simplicity we take $C(y) = 0$.
(2 points)

The general solution of the differential equation is given by

$$(x + y)e^{xy} = K,$$

where K is an arbitrary constant.
(1 point)

Solution of problem 3 (4 + 7 + 9 = 20 points)

(a) The matrix function $Y : \mathbb{R} \to \mathbb{R}^{n\times n}$ is called a fundamental matrix for $(*)$ if the following two conditions are satisfied:

(i) Each column of $Y(t)$ satisfies the homogenous equation $\mathbf{y}' = A(t)\mathbf{y}$. (Equivalent: the matrix $Y(t)$ satisfies $Y' = A(t)Y$.)
(2 points)

(ii) The columns of $Y(t)$ are linearly independent. (Equivalent: the matrix $\det Y(t) \neq 0$.)
(2 points)

(b) If $Y(t)$ is a fundamental matrix for $(*)$, then we can find a solution of the inhomogeneous differential equation by variation of constants. Let $\mathbf{y}_p(t) = Y(t)\mathbf{v}(t)$, then on the one hand we have

$$\mathbf{y}_p' = Y'\mathbf{v} + Y\mathbf{v}' = AY\mathbf{v} + Y\mathbf{v}'.$$

(2 points)

On the other hand we want to have

$$\mathbf{y}_p' = A\mathbf{y}_p + \mathbf{b} = AY\mathbf{v} + \mathbf{b}$$

(2 points)

Therefore we have

$$Y\mathbf{v}' = \mathbf{b} \quad \Rightarrow \quad \mathbf{v}(t) = \int_\tau^t Y(s)^{-1}\mathbf{b}(s)\,ds,$$

where we have chosen a specific antiderivative which satisfies $\mathbf{v}(\tau) = \mathbf{0}$.
(2 points)

The general solution of the differential equation is

$$\mathbf{y}(t) = Y(t)\mathbf{c} + Y(t)\int_\tau^t Y(s)^{-1}\mathbf{b}(s)\,ds,$$

where $\mathbf{c} \in \mathbb{R}^n$ is an arbitrary vector. From the initial condition $\mathbf{y}(\tau) = \boldsymbol{\eta}$ it follows that $\mathbf{c} = Y(\tau)^{-1}\boldsymbol{\eta}$.
(1 point)

(c) The eigenvalues of A are given by the roots of the polynomial

$$\det(A - \lambda I) = \begin{bmatrix} -5-\lambda & 15 \\ -3 & 7-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 10 = (\lambda - 1)^2 + 9$$

Therefore, the eigenvalues are given by $\lambda = 1 \pm 3i$.
(3 points)

For $\lambda = 1 + 3i$ we have

$$A - \lambda I = \begin{bmatrix} -6-3i & 15 \\ -3 & 6-3i \end{bmatrix} \sim \begin{bmatrix} -2-i & 5 \\ -1 & 2-i \end{bmatrix}$$

Clearly, $\mathbf{v} = \begin{bmatrix} 5 & 2+i \end{bmatrix}^\top$ is an eigenvector.

(3 points)

This gives the solution

$$\mathbf{y}(t) = \begin{bmatrix} 5 \\ 2+i \end{bmatrix} e^t(\cos(3t) + i\sin(3t)).$$

(1 point)

Hence, a real fundamental matrix is given by

$$Y(t) = \begin{bmatrix} \operatorname{Re}\mathbf{y}(t) & \operatorname{Im}\mathbf{y}(t) \end{bmatrix} = e^t \begin{bmatrix} 5\cos(3t) & 5\sin(3t) \\ 2\cos(3t) - \sin(3t) & \cos(3t) + 2\sin(3t) \end{bmatrix}.$$

(2 points)

Solution of problem 4 (10 + 4 + 6 = 20 points)

(a) If $y, z \in C([0,1])$ and $x \in [0,1]$, then

$$\begin{aligned}|(Ty)(x) - (Tz)(x)| &= \left| \int_0^x f(y(t)) - f(z(t))\,dt \right| \\ &\le \int_0^x |f(y(t)) - f(z(t))|\,dt \\ &\le \int_0^x 2|y(t) - z(t)|\,dt \\ &= \int_0^x 2|y(t) - z(t)|e^{-\alpha t}e^{\alpha t}\,dt.\end{aligned}$$

(4 points)

Since $|y(t) - z(t)|e^{-\alpha t} \le \|y - z\|$ for all $t \in [0,1]$ we get

$$|(Ty)(x) - (Tz)(x)| \le 2\|y - z\| \int_0^a e^{\alpha t}\,dt$$

(3 points)

After multiplication by $e^{-\alpha x}$ we get

$$|(Ty)(x) - (Tz)(x)|e^{-\alpha x} \le 2\|y - z\| \int_0^x e^{\alpha(t-x)}\,dt.$$

(2 points)

Since this inequality holds for all $x \in [0,1]$ we can take the supremum on both sides, which gives

$$\|Ty - Tz\| \le 2L\|y - z\| \quad \text{where} \quad L = \sup_{x\in[0,1]} \int_0^x e^{\alpha(t-x)}\,dt.$$

(1 point)

(b) Let D be a closed, nonempty subset in a Banach space B. Let the operator $T : D \to B$ map D into itself, i.e., $T(D) \subset D$, and assume that T is a contraction: there exists a number $0 < q < 1$ such that

$$\|Tx - Ty\| \le q\|x - y\|, \qquad \forall\, x, y \in D,$$

Then the fixed point equation $Tx = x$ has precisely one solution $\bar{x} \in D$.
(4 points)

Moreover, iterations of T converge to this fixed point:

$$x_0 \in D, \quad x_{n+1} = Tx_n \quad \Rightarrow \quad \lim_{n\to\infty} x_n = \bar{x}.$$

(The last statement is not relevant to this problem.)

(c) The value of L is given by

$$L = \sup_{x\in[0,1]} \frac{1-e^{-\alpha x}}{\alpha} = \frac{1-e^{-\alpha}}{\alpha},$$

where we have used that the function of which we take the supremum is strictly increasing in x.

(3 points)

For applying Banach's fixed point theorem we need to have $\beta L < 1$, or, equivalently,

$$1-e^{-\alpha} < \frac{\alpha}{\beta}.$$

Note that $0 < 1-e^{-\alpha} < 1$ for all $\alpha > 0$. Hence, by taking $\alpha \geq \beta$ the above inequality is satisfied.

(3 points)

Solution of problem 5 (10 points)

Substitution of $u = e^{\lambda t}$ in the homogeneous differential equation gives the following characteristic equation:

$$2\lambda^3 + 6\lambda^2 - 18\lambda + 10 = 0$$

Clearly, $\lambda = 1$ is a root. A long division then gives

$$\frac{2\lambda^3 + 6\lambda^2 - 18\lambda + 10}{\lambda - 1} = 2\lambda^2 + 8\lambda - 10 = 2(\lambda - 1)(\lambda + 5).$$

Therefore, the roots of the characteristic equation are $\lambda = 1$ (with multiplicity 2) and $\lambda = -5$ (with multiplicity 1).

(4 points)

The general solution of the homogeneous equation is given by

$$u(t) = ae^t + bte^t + ce^{-5t},$$

where a, b, and c are arbitrary constants.

(1 point)

As a particular solution we use the Ansatz $u_p = A + Bt + Ct^2$, which gives

$$\begin{aligned} u_p' &= B + 2Ct \\ u_p'' &= 2C \\ u_p''' &= 0 \end{aligned}$$

Substitution in the differential equation gives

$$10A - 18B + 12C + (10B - 36C)t + 10Ct^2 = 16 + 26t - 10t^2.$$

Comparing coefficients gives $A = 1$, $B = -1$, and $C = -1$.

(4 points)

Hence, the general solution of the inhomogeneous differential equation is given by

$$u(t) = ae^t + bte^t + ce^{-5t} + 1 - t - t^2,$$

where a, b, and c are arbitrary constants.

(1 point)

Solution of problem 6 (5 + 5 + 5 = 15 points)

(a) For $\lambda = 0$ the homogeneous differential equation can be solved as follows:

$$xu'' + u' = 0 \quad \Rightarrow \quad (xu')' = 0 \quad \Rightarrow \quad xu' = a \quad \Rightarrow \quad u' = \frac{a}{x} \quad \Rightarrow \quad u = a\log(x) + b.$$

where a and b are arbitrary constants. (Note that we can omit the absolute values in the logarithm since $x > 0$.)

(3 points)

The boundary conditions imply that $a = b = 0$, which means that the homogeneous equation for $\lambda = 0$ only has the trivial solution. Therefore, $\lambda = 0$ is *not* and eigenvalue.

(2 points)

Note. the equation can also be solved by first multiplying by x. Then we get an Euler equation which can be solved by the Ansatz $u = x^p$, which gives $p^2 = 0$. This gives one constant solution; the other can be found using reduction of order, but this gives again the original equation!

(b) The function $u_1(x) = \log(x)$ satisfies the boundary condition $u(1) = 0$, and the function $u_2(x) = \log(x) - 1$ satisfies the boundary condition $u(e) = 0$.

(2 points)

The Wronskian determinant of u_1 and u_2 is given by

$$W = u_1u_2' - u_1'u_2 = \log(x) \cdot \frac{1}{x} - \frac{1}{x} \cdot (\log(x) - 1) = \frac{1}{x}.$$

Since the coefficient of u'' is $p(x) = x$ it follows that $p(\xi)W(\xi) = 1$ so that the Green's function is given by

$$\Gamma(x,\xi) = \begin{cases} \log(\xi)(\log(x) - 1) & \text{if } 1 \leq \xi \leq x \leq e, \\ \log(x)(\log(\xi) - 1) & \text{if } 1 \leq x \leq \xi \leq e. \end{cases}$$

(3 points)

(c) The solution is given by

$$\begin{aligned} u(x) &= \int_1^e \Gamma(x,\xi)f(\xi)\,d\xi \\ &= \int_1^x \Gamma(x,\xi)f(\xi)\,d\xi + \int_x^e \Gamma(x,\xi)f(\xi)\,d\xi \\ &= (\log(x) - 1)\int_1^x \log(\xi)\,d\xi + \log(x)\int_x^e \log(\xi) - 1\,d\xi \\ &= (\log(x) - 1)\big[-\xi + \xi\log(\xi)\big]_1^x + \log(x)\big[-2\xi + \xi\log(\xi)\big]_x^e \\ &= (\log(x) - 1)(1 - x + x\log(x)) + \log(x)(-e + 2x - x\log(x)). \end{aligned}$$

(5 points)